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THE POSSIBILITY OF GYROSCOPIC STABILIZATION OF THE ROTATION OF A SYSTEM OF RIGID BODIES†

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A simple model is considered for a spaceship (SS) of changing configuration, consisting of a rigid body to which two rods are attached by cylindrical hinges. While performing certain technical tasks the spaceship is brought into a state of permanent rotation about an axis. It is shown that, for certain values of the structural parameters, such a rotation can be gyroscopically stabilized. A qualitative analysis of the range of gyroscopic stabilization in parameter space is carried out.

Analyses of the stability of rotation of a spaceship modelled by a system of rigid bodies generally concentrate on the secular stability [1–3]. As a rule, the possibility of uniform rotation "generates" cyclic coordinates, and the stability of rotation of such systems may be associated with gyroscopic stabilization [4]. This interesting phenomenon is all but ignored in spaceship dynamics, though it may prove useful from a practical point of view.‡ On the other hand, one could hardly expect to get clear results when confined to spaceship models of the type used in [1, 2], since they possess high dimensionality and involve many parameters. Any analysis of the prospects of gyroscopic stabilization of the permanent rotation of a system of rigid bodies should start out, therefore, from a simpler model.

1. FORMULATION OF THE PROBLEM.

Let Q be an axially symmetric body fixed at some point O on its axis of symmetry Oz (see Fig. 1). Two identical rigid rods Q_1 and Q_2 of length L and mass m are attached to Q at points O_1 and O_2 ($|OO_1| = |OO_2| = R$) by cylindrical hinges. The points O_1, O and O_2 lie on a single straight line Oy ($Oy \perp Oz$), and the axes of a hinges are parallel to the Ox axis, which, together with Oz and Oy, forms a system of coordinates attached to Q.

The three-body system thus constructed has five degrees of freedom, and its position relative to the fixed axes $OXYZ$ may be described by five generalized coordinates: θ—the angle between the OX axis and the Zx plane, the angle of rotation of the entire system about the Oz axis, a cyclic coordinate; β—the angle between the Ox axis and the XY plane; α—the angle between the Oz axis and the Zx plane (see Fig. 2). Consequently, the Krylov angles α and β determine the orientation of the Oz axis in some fixed system of coordinates OZx_1y_1 which is rotating about the fixed OZ axis at angular velocity $\theta^{\cdot}$. The position of the rods relative to the body Q is defined by angles φ_1 and φ_2 (Fig. 1).

We shall assume throughout that there are no forces applied and that there is no friction at the hinges.

One of the possible motions of the system is, of course, uniform rotation as a single body about the Oz axis, which is fixed in space, at angular velocity $\theta = \omega = \text{const}$. In that case the rods take positions along the axis Oy ($\varphi_1 = \varphi_2 = 0$).

Any formulation of the stability problem for this rotation must take into consideration that the angular momentum vector $\mathbf{G}$ of the "body-rods" system is constant in magnitude and direction. Hence the fixed OZ axis must point along $\mathbf{G}$.

We shall also use the change of variables

$$\varphi = \varphi_1 + \varphi_2, \quad \psi = \varphi_1 - \varphi_2$$

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‡On some manifestations of gyroscopic stabilization see BELETSKII V. V., Applied problems of stability. Preprint No. 121, Moscow, Inst. Prikl. Mat. im M. V. Keldysha Akad. Nauk SSSR, 1, 1990.

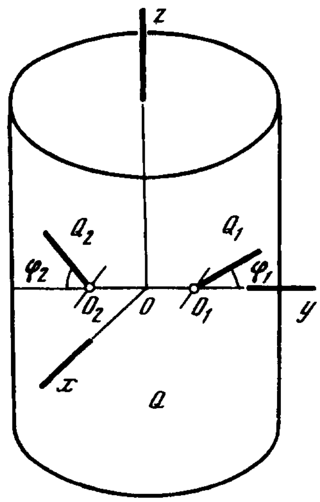

Fig. 1.

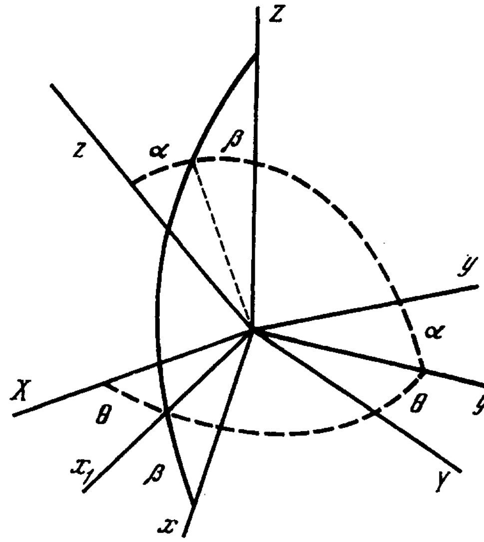

Fig. 2.

as is usual in dealing with such systems, and investigate the stability of rotation of the system to perturbations of the quantities α, β, $\alpha^{\cdot}$, $\beta^{\cdot}$, φ, ψ, $\varphi^{\cdot}$, $\psi^{\cdot}$. Linearizing the equations of motion of the system with respect to these quantities and using matrix notation, we obtain

$$Ax^{\cdot\cdot} + \Gamma\omega x^{\cdot} + K\omega^2 x = 0$$

$$x = \begin{Vmatrix}\alpha\\ \beta\\ \psi\\ \varphi\end{Vmatrix}, \quad A = \|a_{ij}\|, \quad \Gamma = \|g_{ij}\|, \quad K = \|K_{ij}\| \tag{1.1}$$

$$a_{11} = I_1 + \frac{mL^2}{6} + \frac{m(L+2R)^2}{2}, \quad a_{22} = I_1$$

$$a_{33} = a_{44} = \frac{mL^2}{8}, \quad a_{13} = a_{31} = mL(L+2R)/4$$

$$g_{12} = -g_{21} = 2I_1 - I_3, \quad g_{23} = -g_{32} = mL^2/12$$

$$K_{11} = I_3 - I_1 + mL^2/6 + m(L+2R)^2/2, \quad K_{22} = I_3 - I_1$$

$$K_{33} = K_{44} = mL(2L+3R)/12, \qquad K_{13} = K_{31} = mL(2L+3R)/6$$

where I_1 and I_3 are the equatorial and axial moments of inertia of Q and the other coefficients of the matrices A, Γ and K are zeros.

As Eqs (1.1), unlike the equations used in [1, 2], involve generalized coordinates, they can be interpreted [5] as the equations of motion of a certain ("reduced") mechanical system, in which A plays the part of the kinetic energy matrix and which is driven by linear potential (centrifugal) forces with matrix $-K$ and gyroscopic (Coriolis) forces with matrix $-\Gamma$. As we know [4], these observations make it easier to analyse the conditions for the trivial solution of system (1.1) to be stable.

2. THE STABILITY OF RELATIVE EQUILIBRIUM OF A ROD

The last equation of system (1.1) splits off from the other equations, as is quite natural, since symmetric vibrations of the rods do not make the axis of the body oscillate (and vice versa). Hence one condition for stability is $K_{44} > 0$. This condition obviously implies that the relative equilibrium of a rod is stable and formally defines two ranges of admissible parameter values

$$(1)\ L > 0, \quad (2)\ L < -3R/2 \tag{2.1}$$

Although the length of the rod is, of course, necessarily positive, negative values of the parameter L also admit of a perfectly reasonable interpretation. It can be shown that in order to investigate the stability of a configuration in which the rods point "into" the body, i.e. away from the points O_1 and O_2 to the point O ($\varphi_1 = \varphi_2 = \pi$), one need only interchange L and $-L$ in all the formulae. Henceforth, therefore, we shall allow the parameter L to vary from $-\infty$ to $+\infty$, bearing in mind that the domain $L > 0$ corresponds to an "outside" position of the rods and the domain $L < 0$ to an "inside" position.

Thus, the last equation of system (1.1) has a single degree of instability when

$$-3R/2 < L < 0 \tag{2.2}$$

3. ANALYSIS OF POSITIONAL FORCES

We will now consider the subsystem of three equations and single out the corresponding minor K_3 of K

$$K_3 = \begin{Vmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & 0 \\ K_{13} & 0 & K_{33} \end{Vmatrix}$$

The degree of instability of the trivial solution of this subsystem equals the number of negative eigenvalues of K_3. It is clear that the conditions

$$K_{22} = 0, \ K_{33} = 0, \ \Delta = K_{11}K_{33} - K_{13}^2 = 0 \tag{3.1}$$

define certain surfaces in parameter space which separate domains with different degrees of instability.

Dimensionless parameters for the problem are

$$b = \frac{I_1}{I_3} - 1, \qquad l = L/R, \qquad k = \rho R^3 / I_3$$

where ρ is the linear density of a rod ($m = p\,|\,L\,|$).

Figure 3 gives a qualitative representation of a section of the surfaces (3.1) by a plane k = const. The section is projected onto the half-plane $b \geqslant -1/2$, which is physically meaningful, as a certain family of lines. This family divides the half-plane into ten domains, in each of which the signs of the quantities K_{22}, K_{33} and Δ form a different sequence; hence each domain represents a possibly different degree of instability both for the subsystem of the first three equations—N_3, and for the full system (1.1) —N_4 (see Table 1).

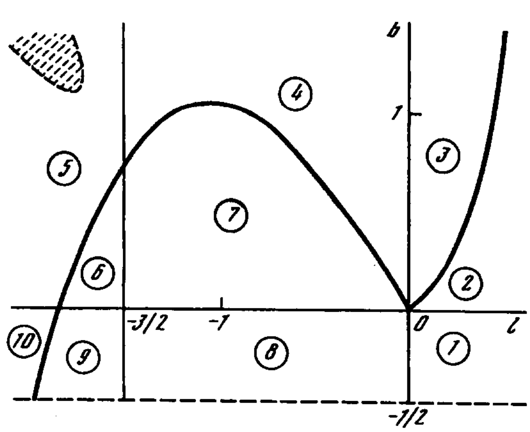

Fig. 3.

Table 1

Domain	K_{22}	K_{33}	Δ	N_3	N_4
1,9	+	+	+	0	0
2.6	–	+	+	1	1
3.5	–	+	–	2	2
4	–	–	+	3	4
7	–	–	–	2	3
8	+	–	–	1	2
10	+	+	–	1	1

First, one should note domains 1 and 9, in which the degree of instability is zero. As follows from the Kelvin–Chetayev theorems [4], addition of gyroscopic forces to the potential forces in these domains does not affect the stability of motion. Moreover, stability is also conserved if one adds dissipative forces depending on $\alpha^{\cdot}$, $\beta^{\cdot}$, $\psi^{\cdot}$, $\varphi^{\cdot}$, which were not included previously—provided the rotation of the system is not terminated by these forces.

Curiously enough, when the rods are "outside" ($L > 0$), their lengths have no effect on stability; while in an "inside" position ($L < 0$) the rod length is constrained to remain between certain non-zero values.

A decrease in the value of k "shifts" the right boundary of domain 3 to the right and the left boundary of domains 6 and 9 to the left.

4. GYROSCOPIC STABILIZATION

It is well known that the equilibrium of the reduced system, which is unstable under the action of several potential forces, can be stabilized by gyroscopic forces only in domains where the degree of instability is even. However, domains 4 and 8 must immediately be dismissed, since gyroscopic forces with matrix Γ have no effect on the nature of symmetric vibrations of the rods, as described by the coordinate φ (see the last equation of system (1.1)).

Thus, the property of gyroscopic stabilization may appear only in domains 3 and 5. To observe it one must analyse the roots of the secular equation of system (1.1) or, more precisely, of the subsystem of its first three equations, and determine in what parts of domains 3 and 5 the roots are purely imaginary. It is not particularly difficult to derive algebraic conditions for the roots to have these properties. However, these conditions, if expressed explicitly in terms of the parameters 1, b and k, are extremely cumbersome and will therefore not be presented here.

Numerical computations carried out using the above formulae have the qualitative result that the stability conditions are satisfied, first, in domains 1 and 9, as might have been expected. Second, such conditions are satisfied in practically all of domain 3, i.e. gyroscopic stabilization is possible in this domain. As to domain 5, the possibility appears only in the small part shown hatched in Fig. 3.

Some curious features are worthy of note. First, in an "outside" position of the rods, gyroscopic stabilization is achieved only provided the rod length has an upper limit (domain 3). In an "inside" position, conversely, the rod length must have a lower limit (domain 5). Second, if the rods are sufficiently long (near the right boundary for domain 3 and sufficiently far to the left for domain 5), the system rotates about the mean axis of inertia for the unperturbed form ($K_{22} < 0$, but $K_{11} > 0$).

5. REDUCTION OF THE SYSTEM

The system being studied here has an interesting property.

Let G_1 and G_2 denote the projections of the angular momentum $\mathbf{G}$ on to the Ox_1 and Oy_1 axes. It follows from the theorem on the variation of $\mathbf{G}$ that

$$G_1^{\cdot} - \omega G_2 = 0, \quad G_2^{\cdot} + \omega G_1 = 0 \tag{5.1}$$

It can be verified that

$$G_1 = a_{11}\alpha^{\cdot} + a_{13}\psi^{\cdot} + k_{22}\omega\beta, \quad G_2 = a_{22}\beta^{\cdot} + k_{11}\omega\alpha - k_{33}\omega\psi$$

and the first two equations of system (1.1) are identical with (5.1). Equations (5.1) have two obvious particular integrals

$$G_1(t) \equiv 0, \quad G_2(t) \equiv 0 \tag{5.2}$$

This face is precisely what was actually exploited above when the fixed Oz axis was chosen, but up to this point it has not been utilized. The integrals (5.2) can be used to reduce the order of the dynamical system (1.1).

For example, if there are no rods ($m = 0$) the coordinate α (or β) satisfies the equation of harmonic oscillations

$$I_1^2\alpha^{\cdot\cdot} + \omega^2(I_3 - I_1)^2\alpha = 0$$

whether the body is "oblate" ($I_3 > I_1$) or "prolate" ($I_3 > I_1$)—an indication of the gyroscopic nature of the stabilization of the body's rotation about the major axis of its inertia ellipsoid.

The use of particular integrals like (5.2) in the general case may alter the type of stability problem to be solved, since, formally speaking, this approach replaces the stability problem in all of phase space by the analogous problem on a certain manifold. In the case considered here, however, the two approaches are equivalent. The reduction only eliminates two of the natural frequencies of system (1.1), namely, $\pm\omega$, corresponding to system (5.1).

This equivalence has made it possible, in particular, to use (5.2) to eliminate φ and $\varphi^{\cdot}$ from the first two equations of system (1.1). The algebraic conditions for the stability of the trivial solution of the reduced system are also rather unwieldy, but numerical computations carried out with them have yielded the same results as for the full system.

6. CONCLUSION

Thus, our analysis of a rotating many-body system has shown that, in a certain range of parameters, gyroscopic stabilization may appear. Of course, this stabilization may be adversely affected by dissipative forces such as friction in the springs. However, if the latter are relatively small, one can expect the use of active means of control in a structure with parameters, say, in domain 3, to impose less stringent demands on control resources than in domain 2. Hence, when dealing with applied problems of the dynamics of a spaceship, it may prove useful to look for the possibility of gyroscopic stabilization.

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